Int J Theor Phys (2007) 46: 2494–2512
DOI 10.1007/s10773-007-9364-8

Conceptual Unification of Gravity and Quanta

Michael Heller · Leszek Pysiak · Wiesław Sasin

Published online: 15 May 2007

Abstract We present a model unifying general relativity and quantum mechanics. The model is based on the (noncommutative) algebra $\mathcal{A}$ on the groupoid $\Gamma = E \times G$ where E is the total space of the frame bundle over spacetime, and G the Lorentz group. The differential geometry, based on derivations of $\mathcal{A}$, is constructed. The eigenvalue equation for the Einstein operator plays the role of the generalized Einstein's equation. The algebra $\mathcal{A}$, when suitably represented in a bundle of Hilbert spaces, is a von Neumann algebra $\mathcal{M}$ of random operators representing the quantum sector of the model. The Tomita–Takesaki theorem allows us to define the dynamics of random operators which depends on the state φ. The same state defines the noncommutative probability measure (in the sense of Voiculescu's free probability theory). Moreover, the state φ satisfies the Kubo–Martin–Schwinger (KMS) condition, and can be interpreted as describing a generalized equilibrium state. By suitably averaging elements of the algebra $\mathcal{A}$, one recovers the standard geometry of spacetime. We show that any act of measurement, performed at a given spacetime point, makes the model to collapse to the standard quantum mechanics (on the group G). As an example we compute the noncommutative version of the closed Friedman world model. Generalized eigenvalues of the Einstein operator produce the correct components of the energy-momentum tensor. Dynamics of random operators does not "feel" singularities.

M. Heller
Vatican Observatory, 00120 Vatican, Vatican

M. Heller (✉)
ul. Powstańców Warszawy 13/94, 33-110 Tarnów, Poland
e-mail: mheller@wsd.tarnow.pl

L. Pysiak · W. Sasin
Department of Mathematics and Information Science, Warsaw University of Technology, Plac Politechniki 1, 00-661 Warsaw, Poland

1 Introduction

One of the driving forces of scientific progress is the evolution of concepts, and concepts evolve when they are involved in solving problems. Currently, the main problem of theoretical physics is to find a sufficiently rich mathematical structure which, when suitably interpreted, would contain in itself (as some "limiting cases") physics of gravity and physics of quanta. It is rather obvious that when this goal is finally reached, it will induce a radical conceptual revolution. In a series of works [17, 18, 20–23]) we have proposed a model, based on noncommutative geometry, unifying general relativity and quantum mechanics (with the perspective of including quantum field theory). We think that the main attractiveness of this model is its conceptual structure firmly based on its mathematical architecture. The main idea of the model consists in exploring a noncommutative algebra $\mathcal{A}$, defined on a transformation groupoid Γ which is given by the action of a group (typically the Lorentz group) on the frame bundle (E, π_M, M) over spacetime M. The geometry of M (physics of gravity) can be recovered by suitably averaging elements of $\mathcal{A}$, and quantum sector of the model is obtained by representing the algebra $\mathcal{A}$ on a family of Hilbert spaces associated with the groupoid Γ. Our approach differs from that of Connes [6] and the authors following him (e.g., [3–5, 25, 26, 29, 35, 37]) in that we explore the structure of the groupoid Γ and base our construction of the noncommutative differential algebra on $\mathcal{A}$ and the (sub)module of its derivations (similarly to the approach developed by [10]; see also [11]] whereas Connes does this on the representation of the corresponding algebra on a Hilbert space, and he uses differential forms rather than derivations. We go to the representation of $\mathcal{A}$ only to recover the quantum sector of our model.

In the present paper, we further develop our model, both its mathematical and conceptual aspects. We show how strongly these aspects interact with each other. To make the paper self-contained, new results (indicated below) are presented in a broader context of the model's structure. In Sect. 2, we briefly present the groupoid $\Gamma = E \times G$ (in the present paper G is a noncompact group) and the noncommutative algebra $\mathcal{A}$ of smooth compactly supported, complex valued functions on Γ with convolution as multiplication. The groupoid can be regarded as a space of generalized symmetries of our model. The differential geometry of the groupoid Γ is based on the algebra $\mathcal{A}$ and its derivations. Derivations are classified into three types: horizontal V_1, vertical V_2 and inner V_3 [23]. The pair $(\mathcal{A}, V)$, where V is a subset of the module $\mathrm{Der}(\mathcal{A})$ of all derivations of the algebra $\mathcal{A}$, is called a differential algebra. We introduce it in Sect. 3. The gravitational sector of the model is presented in Sect. 4. It is based on the differential algebra $(\mathcal{A}, V)$ where $V = V_1 \oplus V_2$. We first construct the corresponding differential geometry (connection, curvature, Einstein operator), and then we postulate that the eigenvalue equation for the Einstein operator should play the role of a generalized Einstein's equation (no energy-momentum tensor is assumed). This is an important modification with respect to [23]; its best justification being the result obtained in Sect. 5, where the components of this equation are computed for the closed Friedman world model. It turns out (by comparison with the usual Friedman model) that the (generalized) eigenvalues of the Einstein operator should be interpreted as matter sources. A suitable equation of state turns out to be encoded in a relationship between different eigenvalues. This example also shows that the groupoid Γ and the noncommutative algebra $\mathcal{A}$ are essential elements of the model; without them this result could not be obtained. In Sect. 6, we present the quantum sector of the model. The algebra $\mathcal{A}$, when suitably represented in a bundle of Hilbert spaces, is a von Neumann algebra $\mathcal{M}$ of random operators (the von Neumann algebra of the groupoid Γ). The Tomita–Takesaki theorem, applied to this algebra, allows us to define the dynamics of random operators which depends on a state φ on $\mathcal{M}$. The pair $(\mathcal{M}, \varphi)$ can thus be regarded as a "dynamic object" of our model. In Sect. 7, we summarize the above results by defining a dynamical system for our model. It consists of two equations: the eigenvalue equation for the Einstein operator and the dynamical equation of the Tomita–Takesaki theorem. The first of these equations corresponds to the differential algebra $(\mathcal{A}, V_1 \oplus V_2)$, the second is related, via a suitable representation, to the differential algebra $(\mathcal{A}, V_3)$. In Sect. 8, we discuss some dynamical properties of our model. If φ, appearing in the modular evolution, is a faithful and normal state, it also defines the noncommutative probability measure [38, 39]. Thus the pair $(\mathcal{M}, \varphi)$ is both a "dynamic object" and a "probabilistic object" of the model. For the full discussion of these properties one should refer to [22, 23]; here a new element has been added: if the state φ satisfies the Kubo–Martin–Schwinger (KMS) condition, it can be interpreted as describing a generalized equilibrium state [27, 34]. Therefore, on the fundamental level, dynamics, probability and at least some thermodynamic properties are encoded in the same mathematical structure. In Sect. 9, we return to the example of the noncommutative version of the closed Friedman universe, and explore its quantum sector. The most intriguing result is that the random dynamics on the fundamental (Planck) level does not "feel" singularities. They emerge, together with spacetime when the noncommutative regime changes into the usual commutative evolution. In Sect. 10, we show how to obtain general relativity and quantum mechanics from our model as its limiting cases. A few remarks, in Sect. 11, concerning perspectives of the model close the paper.

2 Noncommutative Generalization of Spacetime

In our first encounter with the special theory of relativity we were not immediately exposed to the spacetime geometry or to some other abstract mathematical structures but rather we were instructed how to change from one inertial reference frame to another inertial reference frame with the help of a Lorentz transformation. In this sense, the set of pairs of reference frames and elements of the Lorentz group transforming these frames into one another forms a natural setting for the special theory of relativity. If a suitable care is applied, the same procedure could be extended to general relativity. These considerations lead to the following construction.

Let $\pi_M : E \to M$ be a frame bundle over spacetime M with the structural group $G = SO_0(3,1)$. A fibre $E_x = \pi_M^{-1}(x)$ over $x \in M$ is the set of local reference frames attached to the point x. For any pair of such frames $p, q \in E_x$ there exists $g \in G$ such that $p = qg$. We see that G acts on E, $E \times G \to E$, along the fibres. This allows us to construct the Cartesian product

$$\Gamma = E \times G = \{\gamma = (p, g) : p \in E, g \in G\},$$

two elements of which, $\gamma_1 = (p_1, g_1)$ and $\gamma_2 = (p_2, g_2)$, can be composed (multiplied) if $p_2 = p_1 g_1$ to give $\gamma_1 \circ \gamma_2 = (p_1, g_1 g_2)$. The inverse of $\gamma = (p, g)$ is $\gamma^{-1} = (pg, g^{-1})$. There are two mappings: $d(p, g) = p$ and $r(p, g) = pg$, called the *source* mapping and the *target* mapping, respectively. The *set of units* is defined to be $\Gamma^{(0)} = \{(p, e) : p \in E\}$ where e is the unit of G. If some natural conditions are satisfied, Γ is called *groupoid* (for definition see [20, 31, 32]). If this purely algebraic construction is equipped with the smoothness structure, it is called a *smooth* or *Lie groupoid*.

The above described groupoid Γ implements the idea of a space, the elements of which consist of two reference frames $d(p, g) = p$ and $r(p, g) = pg$, and the element g of the Lorentz group G transforming p into $q = pg$. We refer to Γ as to the *transformation* *groupoid*. The same idea can be implemented by specifying only two reference frames p_1 and p_2 attached to the same point x of M. We thus define

$$\Gamma_1 = \{(p_1, p_2) \in E \times E : \pi_M(p_1) = \pi_M(p_2) = x \in M\}$$

with the composition law: $(p_1, p_2) \circ (p_2, p_3) = (p_1, p_3)$, $p_1, p_2, p_3 \in E$. It is a groupoid called *groupoid of pairs*. In fact, Γ and Γ_1 are isomorphic as groupoids [23].

As it is well known, the geometry of spacetime M can be reconstructed in terms of the algebra $C^\infty(M)$ of smooth functions on M. Moreover, also Einstein's equations can be defined in terms of this algebra [12–14]. The natural idea would be to apply the same strategy to the space Γ (or Γ_1). It turns out that to obtain the case interesting from both mathematical and physical points of view, we should consider a noncommutative algebra $\mathcal{A}$ on the groupoid Γ. A commutative algebra $(\mathcal{A}, \cdot)$ of smooth, complex, compactly supported functions on Γ with the usual pointwise multiplication would give us again a classical geometry. To obtain a noncommutative algebra we replace the usual pointwise multiplication with the convolution: if $f_1, f_2 \in \mathcal{A}$ then

$$(f_1 * f_2)(\gamma) = \int_{\Gamma_{d(\gamma)}} f_1(\gamma_1) f_2(\gamma_1^{-1}\gamma) d\gamma_1$$

where the integration is over all elements $\gamma \in \Gamma$ beginning at $p = d(\gamma)$ which is denoted by $\Gamma_{d(\gamma)}$. Let us notice that convolution algebras play the crucial role in harmonic analysis and in representation theory, both in the case of groups and groupoids. The algebra $(\mathcal{A}, *)$, being now noncommutative, is nonlocal. It has no maximal ideals which would correspond to points and their neighborhoods in Γ. The groupoid Γ is replaced by its noncommutative version, i.e., by the "virtual space" corresponding to the algebra $(\mathcal{A}, *)$. Keeping this in mind we shall denote this algebra by $\mathcal{A} = (C_c^\infty(\Gamma, \mathbf{C}), *)$. As we shall see in the following sections, this algebra is rich enough to contain it itself both relativistic and quantum structures.

If we chose the groupoid Γ_1 rather than the groupoid Γ, we should define the algebra $\mathcal{A}_1$ on Γ_1 via the isomorphism $J : \mathcal{A}_1 \to \mathcal{A}$ given by $J(f)(\gamma) = f(p, pg)$ for $f \in \mathcal{A}_1$ and $\gamma = (p, g)$ [23].

3 Differential Algebra

The first thing we must ensure is that the noncommutative geometry based on the algebra $\mathcal{A}$ should allow us to recover the usual (noncommutative) spacetime geometry as a special case. The natural way of doing this would be by restricting the algebra $\mathcal{A}$ to its center $\mathcal{Z}(\mathcal{A})$ (i.e., to the subset of $\mathcal{A}$ consisting of all these elements that commute with all elements of $\mathcal{A}$). Unfortunately, $\mathcal{Z}(\mathcal{A}) = \{0\}$. It turns out, however, that the lifting of the algebra $C^\infty(M)$ to the total space E of the frame bundle over M, i.e., the set $Z = \pi_M^*(C^\infty(M))$ (which is, of course isomorphic with $C^\infty(M)$), can be regarded as an "outer center" of the algebra $\mathcal{A}$. To be more precise, although the functions belonging to Z are not compactly supported, we can define their action on the algebra $\mathcal{A}$, $\alpha : Z \times \mathcal{A} \to \mathcal{A}$, by

$$\alpha(f, a)(p, g) = f(p)a(p, g),$$

$f \in Z, a \in \mathcal{A}$. We see that the algebra $\mathcal{A}$ is a module over $Z = \pi_M^*(C^\infty(M))$. This fact allows us to develop a noncommutative geometry based on the algebra $\mathcal{A}$ which will be a true generalization of the usual spacetime geometry.

One can base the noncommutative geometry either on differential forms defined in terms of the algebra $\mathcal{A}$ [3, 6, 25, 26, 35], or in terms of derivations of this algebra [10, 11, 24]. The first method is more common, but the second method is closer to the usual way of doing (commutative) differential geometry. In our case, there is plenty of derivations, and the second method seems more appropriate.

A *derivation* of the algebra $\mathcal{A}$ is a linear map $v : \mathcal{A} \to \mathcal{A}$ satisfying the Leibniz rule

$$v(a, b) = v(a)b + av(b).$$

It can be thought of as a generalization of the vector field concept. The set of all derivations of the algebra $\mathcal{A}$ will be denoted by $\mathrm{Der}(\mathcal{A})$. It has the algebraic structure of a Z-module.

Let $\bar{X}$ be a vector field on E; we shall write $\bar{X} \in \mathcal{X}(E)$. Let us also assume that $\bar{X}$ is a *right invariant* vector field, i.e.

$$(\mathcal{R}_g)_{*p}\bar{X}(p) = \bar{X}(pg)$$

for every $g \in G$. The *lifting* of $\bar{X}$ to Γ is defined to be

$$\bar{\bar{X}}(p, g) = (\iota_g)_{*p}\bar{X}$$

where the inclusion $\iota_g : E \hookrightarrow E \times G$ is given by $\iota_g(p) = (p, g)$. It can be shown that the lifting of a right invariant vector field $\bar{X} \in \mathcal{X}(E)$ to Γ is a derivation of the algebra $\mathcal{A}$ [23].

Let $\bar{X} \in \mathcal{X}(E)$ be a right invariant vector field. If it satisfies the condition $(\pi_M)_*\bar{X} = 0$ it is said to be a *vertical* vector field. Such vector fields, when lifted to Γ, are derivations of the algebra $\mathcal{A}$ and are called *vertical derivations*.

Let us suppose that a connection is given in the frame bundle $\pi_M : E \to M$ (for details see [23]). With the help of this connection we lift a vector field X on M to E, i.e., $\bar{X}(p) = \sigma(X(\pi_M(p)), \pi_M(p) = x \in M$ where σ is a lifting homomorphism. This vector field is right invariant on E. If we lift it further to Γ

$$\bar{\bar{X}}(p, g) = (\iota_g)_{*p}\bar{X}(p) \in \mathcal{X}(\Gamma),$$

it preserves its right invariance property, and is a derivation of the algebra $\mathcal{A}$. We call it a *horizontal derivation* of $\mathcal{A}$.

The algebra $\mathcal{A}$ has also derivations typical for noncommutative algebras. They are called *inner derivations*, denoted by $\mathrm{Inn}(\mathcal{A})$, and defined to be

$$\mathrm{Inn}(\mathcal{A}) = \{ad(a) : a \in \mathcal{A}\}$$

where $(ad(a))(b) := a * b - b * a$. Of course, for commutative algebras all such derivations vanish. It is important to notice that the mapping $\Phi(a) = ad(a)$, for every $a \in \mathcal{A}$, establishes the isomorphism between the algebra $\mathcal{A}$ and the space $\mathrm{Inn}(\mathcal{A})$ as Z-moduli [20].

By the *differential algebra* we understand a pair $(\mathcal{A}, V)$ where $\mathcal{A}$ is (not necessarily commutative) algebra and $V \subset \mathrm{Der}(\mathcal{A})$ is a (sub)module of its derivations. In Sect. 4, we construct the gravitational sector of our model basing on the differential algebra $(\mathcal{A}, V)$ where $V = V_1 \oplus V_2$ with V_1 and V_2 being horizontal and vertical derivations, respectively. The Z-module $V_3 = \mathrm{Inn}(\mathcal{A})$ is responsible for quantum effects; it is taken into account in Sect. 6.

4 Gravitational Sector

4.1 Geometry

In the present section, we compute the "groupoid geometry" for the case when $V = V_1 \oplus V_2$ and G is a noncompact and semisimple group (which includes the group $SO_0(3,1)$); the case with a finite group G was treated in [21]. As the metric $\mathcal{G} : V \times V \to Z$ we choose

$$\mathcal{G}(u, v) = \bar{g}(u_1, v_1) + \bar{k}(u_2, v_2)$$

where $u_1, v_1 \in V_1, u_2, v_2 \in V_2$. The metric $\bar{g}$ is evidently the lifting of the metric g on space-time M, i.e.,

$$\bar{g}(u_1, v_1) = \pi_M^*(g(X, Y))$$

where $X, Y \in \mathcal{X}(M)$. We assume that the metric $\bar{k}$ is of the Killing type. In principle, more general metrics than $\mathcal{G}$ could also be considered.

The next step in our construction is to define preconnection by the Koszul formula

$$\begin{aligned}(\nabla_u^* v)w = \frac{1}{2}[&u(\mathcal{G}(v, w)) + v(\mathcal{G}(u, w)) - w(\mathcal{G}(u, v)) \\ &+ \mathcal{G}(w, [u, v]) + \mathcal{G}(v, [w, u]) - \mathcal{G}(u, [v, w]).\end{aligned}$$

In [23] we have proved that if V is a Z-module of derivations of an algebra $(\mathcal{A}, *)$ such that $V(Z) = \{0\}$ then, for every symmetric nondegenerate tensor $g : V \times V \to Z$, there exists exactly one connection g-consistent with the preconnection ∇^* which is given by

$$\nabla_u v = \frac{1}{2}[u, v].$$

We assume that, for V_2, the metric is of the Killing form

$$g(u, v) = \mathrm{Tr}(u \circ v)$$

(the g-consistency condition is clearly satisfied). For the group G (which in our case is semisimple) the Killing form is

$$\mathcal{B}(V, W) = \mathrm{Tr}(ad(V) \circ ad(W))$$

where V, W are elements of the Lie algebra $\underline{g}$ of the group G. The Killing form $\mathcal{B}$ is nondegenerate. Since the tangent space to any fiber $E_x, x \in M$ at a fixed point $p \in E$, is isomorphic to $\underline{g}$, each invariant vertical vector field $\bar{X}$ can be represented by a $\underline{g}$-valued function on E, and one can prove that $\mathcal{B}(\bar{X}(p), \bar{Y}(p))$ depends only on $\pi_M(p) \in M$. Therefore, the metric $\bar{k} : V_2 \times V_2 \to Z$ is given by

$$\bar{k}(\bar{\bar{X}}, \bar{\bar{Y}}) = \mathcal{B}(\bar{X}(p), \bar{Y}(p)).$$

The trace for the algebra $\mathcal{A}_1$ (which is isomorphic to the algebra $\mathcal{A}$) is given by

$$\mathrm{Tr}(a)(p) = \int_G a(pg, pg)dg.$$

It is clear that $\mathrm{Tr}(a) \in Z$.

The *curvature*, for all V_i, $i = 1, 2$, is defined in the usual way

$$\overset{i}{R}(u, v)w = \overset{i}{\nabla}_u \overset{i}{\nabla}_v w - \overset{i}{\nabla}_v \overset{i}{\nabla}_u w - \overset{i}{\nabla}_{[u,v]} w.$$

For $i = 2$, we readily compute

$$\overset{2}{R}(u, v)w = -\frac{1}{4}[[u, v], w].$$

For $i = 1, 2$ and every endomorphism $T : V_i \to V_i$, there exists the usual trace $\mathrm{Tr}(T) \in Z$, and we can define $\overset{i}{R}_{uw}: V_i \to V_i$ by

$$\overset{i}{R}_{uw}(v) = \overset{i}{R}(u, v)w.$$

Consequently, the *Ricci curvature* is

$$\overset{i}{\mathbf{ric}}(u, w) = \mathrm{Tr}(\overset{i}{R}_{uw}),$$

and the *adjoint Ricci operator* $\overset{i}{\mathcal{R}}: V_i \to V_i$ is given by

$$\overset{i}{\mathbf{ric}}(u, w) = \overset{i}{\mathcal{G}}(\overset{i}{\mathcal{R}}(u), w)$$

where $\overset{1}{\mathcal{G}} = \bar{g}$ and $\overset{2}{\mathcal{G}} = \bar{k}$. If the metric $\overset{i}{\mathcal{G}}$ is nondegenerate, there exists the unique $\overset{i}{\mathcal{R}}$ satisfying the above equation for every $w \in V_i$.

The *curvature scalar* is

$$\overset{i}{r} = \mathrm{Tr}(\overset{i}{\mathcal{R}}).$$

For V_2 (for which the usual trace exists) we compute

$$\overset{2}{\mathbf{ric}}(u, w) = \frac{1}{4}\bar{k}(u, w)$$

for every $u, w \in V_2$.

4.2 Generalized Einstein Equation

In the present paper we postulate that the *generalized Einstein equation* should have the form of the eigenvalue equation for the Einstein operator $\mathbf{G} := \mathcal{R} - \frac{1}{2} r \,\mathrm{id}_V$ where $r = \mathrm{Tr}\,\mathcal{R}$. Let us notice that we do not assume *a priori* energy-momentum tensor in any form. The motivation for this assumption is that on the fundamental level we expect to have a "pure pregeometry", and the "matter content" should be somehow produced from it at a later stage. As we shall see in the next section, this is indeed the case, at least for the noncommutative version of the closed Friedman world model.

The eigenvalue equation for the Einstein operator is

$$\mathbf{G} - \tau \,\mathrm{id}_V = 0 \tag{1}$$

where $\tau \in Z$ and $v \in V$; τ will be called a generalized eigenvalue of the operator $\mathbf{G}$ (generalized, because it is a function rather than a number).

If $v \in V_1$, (1) reflects essentially the geometry of space-time M.

If we assume that the metric $\bar{k}$ is of the Killing type then for $v \in V_2$ and $G = SO_0(3,1)$ the dimension of the Z-module V_2 is equal to the dimension of the Lie algebra of G which is 6-dimensional. Consequently, $\mathrm{Tr}(\mathrm{id}_{V_2}) = 6$. Since, in this case, $\mathcal{R} = \frac{1}{4}\mathrm{id}_{V_2}$, we have $r = \frac{3}{2}$, and the Einstein equation assumes the form

$$\left(\tau + \frac{1}{2}\right)(v) = 0.$$

We see that for $\tau = -\frac{1}{2}$ every $v \in V_2$ solves this equation, and for $\tau \neq -\frac{1}{2}$ there is only the trivial solution.

5 Noncommutative Closed Friedman Universe

As a simple example let us consider the closed Friedman world model. Its spacetime $M = \{(\eta, \chi, \theta, \varphi) : \eta \in (0, T), (\chi, \theta, \varphi) \in S^3\} = (0, T) \times S^3$, where $(0, T) \subset \mathbf{R}$, carries the metric

$$ds^2 = R^2(\eta)(-d\eta^2 + d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)).$$

The initial singularity is characterized by: $R^2(\eta) \to 0$ as $\eta \to 0$, and the final singularity by: $R^2(\eta) \to 0$ as $\eta \to T$.

Let $(\pi_M : E \to M)$ be a frame bundle over M where

$$E = \{(\eta, \chi, \theta, \varphi, \lambda) : (\eta, \chi, \theta, \varphi) \in M, \lambda \in \mathbf{R}\} = M \times \mathbf{R}.$$

The structural group of the frame bundle is

$$G = \left\{ \begin{pmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, t \in \mathbf{R} \right\},$$

To have the orthonormal frames we make the transformation $\partial_\mu \to \frac{1}{R(\eta)}\partial_\mu$. This group of "Lorentz rotations" ([2], p. 22) has been chosen by us because, in spite of its simplicity, it gives an insight into many aspects of the general case (see [9], Sect. 3).

The space of the pair groupoid is given by

$$\Gamma = \{(\eta, \chi, \theta, \varphi, \lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in \mathbf{R}\}.$$

If $a, b \in \mathcal{A} = C_c^\infty(\Gamma, \mathbf{C})$ then

$$(a * b)(\eta, \chi, \theta, \varphi, \lambda_1, \lambda_2) = \int_{\mathbf{R}} a(\eta, \chi, \theta, \varphi, \lambda_1, \lambda) b(\eta, \chi, \theta, \varphi, \lambda, \lambda_2) d\lambda.$$

The "outer center" of this algebra is $Z = \{a(\eta, \chi, \theta, \varphi) : (\eta, \chi, \theta, \varphi) \in M\}$. Since the convolution is defined on the groupoid rather than on the group, the algebra $\mathcal{A}$ is noncommutative in spite of the fact that the group $G = \mathbf{R}$ is Abelian.

We consider the Z-submodule $V = V_1 \oplus V_2$ of horizontal derivations and vertical derivations of the algebra $\mathcal{A}$. The metric on V is

$$ds^2 = -R^2(\eta)d\eta^2 + R^2(\eta)d\chi^2 + R^2(\eta)\sin^2(\chi)d\theta^2 + R^2(\eta)\sin^2(\chi)\sin^2(\theta)d\varphi^2 + d\lambda^2.$$

The Einstein operator is of the form $\mathbf{G} = G^c{}_d = \mathrm{diag}\{B, h, h, h, q\}$ where

$$B = -3\frac{1}{R^2(t)} - 3\frac{R'^2(t)}{R^4(t)}, \qquad h = -\frac{1}{R^2(t)} + \frac{R'^2(t)}{R^4(t)} - 2\frac{R''(t)}{R^3(t)},$$
$$q = -3\frac{1}{R^2(t)} - 3\frac{R''(t)}{R^3(t)}.$$

We assume the field equation in the form of the eigenvalue equation for the Einstein operator $\mathbf{G} : V \to V$, i.e.,

$$\mathbf{G}(u) = \tau \cdot u, \tag{2}$$

or in the matrix form

$$\begin{pmatrix} B & 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 & 0 \\ 0 & 0 & h & 0 & 0 \\ 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \tau \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}$$

where $u_1, u_2, u_3, u_4 \in V_1$, $u_5 \in V_2$.

Here $\tau = (\tau_1, \dots, \tau_5)$, where $\tau_i = 1, 2, \dots, 5$, are generalized eigenvalues of $\mathbf{G}$. We find them by solving the equation

$$\det(\mathbf{G} - \tau \cdot \mathbf{I}) = 0. \tag{3}$$

The solutions are

$$\tau_B = -3\frac{1}{R^2(\eta)} - 3\frac{R'^2(\eta)}{R^4(\eta)}, \tag{4}$$

$$\tau_h = -\frac{1}{R^2(\eta)} + \frac{R'^2(\eta)}{R^4(\eta)} - 2\frac{R''(\eta)}{R^3(\eta)}, \tag{5}$$

$$\tau_q = -3\frac{1}{R^2(\eta)} - 3\frac{R''(\eta)}{R^3(\eta)}. \tag{6}$$

The eigenvectors corresponding to these generalized eigenvalues τ_i form the submodules W_i, $i = 1, \dots, 5$ of V

$$V_1 \oplus V_2 = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5,$$

or

$$V_1 \oplus V_2 = W_B \oplus W_h \oplus W_q$$

where $W_B = W_1$ is a 1-dimensional submodule corresponding to the generalized eigenvalue τ_B, $W_h = W_2 \oplus W_3 \oplus W_4$ is a 3-dimensional submodule corresponding to the generalized eigenvalue τ_h, and $W_q = W_5$ is a 1-dimensional submodule corresponding to the generalized eigenvalue τ_q.

Let us notice that $\mathbf{G}$ is a homothety on W_i with the factor τ_i.

By comparing equations (4) and (5) with the components of the perfect fluid energy-momentum tensor for the Friedman world model, we easily identify

$$\tau_B = 8\pi G\rho(\eta),$$

$$\tau_h = -8\pi G p(\eta)$$

where G is the Newtonian gravitational constant, ρ and p are density and stress functions, respectively, and the velocity of light $c = 1$. If we denote

$$T_0^0 = \frac{\tau_B}{8\pi G},$$

$$T_k^i = -\frac{\tau_h}{8\pi G}\delta_k^i, \quad i, k = 1, 2, 3,$$

we obtain the components of the energy-momentum tensor T_ν^μ, $\mu, \nu = 0, 1, 2, 3$ as generalized eigenvalues of the Einstein operator $\mathbf{G}$ corresponding to the submodules W_B and W_h, respectively.

And what about (6)? We easily verify that

$$\tau_q = 4\pi G(\rho(\eta) - 3p(\eta)) \tag{7}$$

which leads to

$$\tau_q = \frac{1}{2}\tau_B + \frac{3}{2}\tau_h. \tag{8}$$

It can be easily seen that τ_q is the trace of T_ν^μ. Let us also notice that (7) is related to the equation of state for the Friedman model. Indeed,

- if $\tau_q = 4\pi G\rho$ then we have the equation of state for dust $p = 0$;
- if $\tau_q = 0$ then we have the equation of state for radiation $p = (1/3)\rho$;
- if $\tau_q = -16\pi G p$ then we have Zeldovitch's stiff equation of state $p = -\rho$.

As we can see, the remaining generalized eigenvalue is responsible for the equation of state. From the mathematical point of view, any formula satisfying (8) can serve as an equation of state, although only some of them have a physical meaning.

The usual Einstein equations are obtained in a straightforward way

$$\mathbf{G}|_{W_B \oplus W_h} = 8\pi G \cdot T,$$

where $T = T_\nu^\mu$, which read

$$8\pi G\rho(\eta) = -3\frac{1}{R^2(\eta)} - 3\frac{R'^2(\eta)}{R^4(\eta)}, \tag{9}$$

$$8\pi G p(\eta) = \frac{1}{R^2(\eta)} - \frac{R'^2(\eta)}{R^4(\eta)} + 2\frac{R''(\eta)}{R^3(\eta)}. \tag{10}$$

It is straightforward to verify that solving equations (9) and (10) for the above equations of state, we obtain the well known solutions

- $R(\eta) = \frac{4\pi m}{3}(1-\cos(\eta))$, where $m = \rho(\eta)R^3(\eta) = \text{const}$, for dust;
- $R(\eta) = \sqrt{\frac{8\pi GM}{3}}\sin(\eta)$, where $M = \rho(\eta)R^4(\eta) = \text{const}$, for radiation; and
- $R(\eta) = 2G^{1/4}(\frac{2\pi}{3})^{1/4}N^{1/4}\sqrt{\frac{\tan(\eta}{1+\tan^2(\eta)}}$, where $N = \rho(\eta)R^6(\eta) = \text{const}$, for Zeldovitch's stiff matter.

We have started with field equation (2) understood as the eigenvalue equation for the Einstein operator, and by solving the eigenvalue problem we were able to produce the perfect fluid energy momentum tensor. No matter source has been assumed *a priori*. In this sense, we can say that in the noncommutative closed Friedman model geometry generates matter. It was an old Wheeler's idea to produce "matter out of pregeometry" [40]; the latter being "a combination of hope and need, of philosophy and physics and mathematics and logic" ([28], p. 1203). The effect presented above can be regarded as a step towards the implementation of this idea in the context of a concrete mathematical model.

In Sect. 9 we discuss some aspects of the quantum sector of the closed Friedman model.

6 Quantum Sector

6.1 Algebra of Random Operators

The quantum sector of our model can be extracted from the groupoid algebra $\mathcal{A}$ with the help of its regular representation in the Hilbert space $\mathcal{H}^p = L^2(\Gamma^p)$, for every $p \in E$ (Γ^p being the set of all elements of Γ ending at p),

$$\pi_p : \mathcal{A} \to \mathcal{B}(\mathcal{H}^p),$$

where $\mathcal{B}(\mathcal{H}^p)$ is the algebra of bounded operators on the Hilbert space $\mathcal{H}^p$. The representation π_p is given by

$$(\pi_p(a)\psi)(\gamma) = \int_{\Gamma_{d(\gamma)}} a(\gamma_1)\psi(\gamma_1^{-1}\circ\gamma)d\gamma_1$$

where $a \in \mathcal{A}$, $\psi \in \mathcal{H}^p$, $\gamma, \gamma_1 \in \Gamma$. Here the Haar measure on the group G, transferred to each fiber of Γ, forms a Haar system on Γ.

It is interesting to notice that the quantum sector of our model exhibits strong probabilistic properties from the very beginning (without putting them by hand into the model). We shall show that every $a \in \mathcal{A}$ generates a random operator $r_a = (\pi_p(a))_{p\in E}$, acting on a collection of Hilbert spaces $\{\mathcal{H}^p\}_{p\in E}$ where $\mathcal{H}^p = L^2(\Gamma^p)$.

An operator r_a is a *random operator* if it satisfies the following conditions [6].

(1) If $\xi_p, \eta_p \in \mathcal{H}^p$ then the function $E \to \mathbf{C}$, given by

$$E \ni p \mapsto (r_a\xi_p, \eta_p),$$

$a \in \mathcal{A}$, is measurable in the usual sense (i.e., with respect to the usual manifold measure on E).

(2) The operator r_a is *bounded*, i.e., $\|r_a\| < \infty$ where

$$\|r_a\| = \text{ess sup}\,\|\pi_p(a)\|.$$

Here "ess sup" denotes essential supremum, i.e., supremum modulo zero measure sets.

In our case, both these conditions are satisfied. Let us also notice that $\pi_p(a)$, for every $p \in E$, is a bounded operator on $\mathcal{H}^p$.

There exist the isomorphisms $I_p : L^2(G) \to \mathcal{H}^p$, for every $p \in E$, given by

$$(I_p\psi)(pg^{-1}, g) = \psi(g)$$

which can be used to establish the relationship between random operators and operators on $\mathcal{H}^p$. These isomorphisms will play an important role in our further analysis.

Let us denote by $\mathcal{M}_0$ the algebra of equivalence classes (modulo equality almost everywhere) of bounded random operators $r_a, a \in \mathcal{A}$, and let us define $\mathcal{M} = \mathcal{M}_0''$ where $\mathcal{M}_0''$ denotes the double commutant of $\mathcal{M}_0$. The algebra $\mathcal{M}$ is a von Neumann algebra [6]. We shall call it the *von Neumann algebra of the groupoid Γ*.

As well known, the work of Segal, Kastler, Haag, Gelfand an others has developed an algebraic description of quantum systems (with both finite and infinite number of degrees of freedom). It consists of the following main ingredients: (1) an abstract C^*-algebra encoding, among others, observables of the system and its statistical properties, (2) automorphisms of this algebra encoding the dynamics of the system and its symmetries, and (3) a state functional defining a probability measure on observables [1]. It can be shown that $\mathcal{M}$ itself is a C^*-algebra [30] and, as we shall see in the following subsections, it satisfies all the above requirements. Therefore, the algebra $\mathcal{M}$ is a mathematical structure that can be interpreted as a true generalization of the usual quantum theory in its algebraic formulation.

6.2 Noncommutative Dynamics

The mathematical structure of our model is encoded in the differential algebra $(\mathcal{A}, \text{Der}(\mathcal{A}))$. As we have seen in Sect. 4, the field equation of the gravitational sector was obtained by considering the Z-submodule $V = V_1 \oplus V_2 \subset \text{Der}(\mathcal{A})$; in the present subsection we consider the Z-submodule $V_3 = \text{Inn}(\mathcal{A})$, and show that this leads to the dynamic equation of the quantum sector of our model.

Let us then consider the differential algebra $(\mathcal{A}, \text{Inn}(\mathcal{A}))$, and let us remember that $\mathcal{A}$ and $\text{Inn}(\mathcal{A})$ are isomorphic as Z-moduli (the isomorphism id given by $a \mapsto \text{ad}_a$, see above Sect. 3). Moreover, every $a \in \mathcal{A}$ generates the random operator $r_a = (\pi_p(a))_{p\in E}$ and, as we have seen, the space $\mathcal{M}_0$ of such operators can be completed to the von Neumann algebra $\mathcal{M}$. (It is interesting to notice that all derivations of any von Neumann algebra are inner [8], pp. 349–357.) In [23] we have shown that the Tomita–Takesaki theorem can be applied to the algebra $\mathcal{M}$ (this algebra is semifinite) to obtain the evolution of random operators (see also [7]). Let us define the Hamiltonian as $H_p^\varphi = \text{Log}\hat{\rho}_p^\varphi$, where $\hat{\rho}(p)$ is a positive, trace class operator in $\mathcal{B}(\mathcal{H}^p)$, and φ is a state on $\mathcal{M}$ defined to be

$$\varphi(A) = \int_M \text{tr}(\hat{\rho}(p)A(p))d\mu(x).$$

Let us notice that the integrated function depends only on $x \in M$. We additionally assume that $\varphi(1) = 1$ [23]. On the strength of the Tomita–Takesaki theorem there exists a one-parameter group of automorphisms σ_t^φ, called *modular group*,

$$\sigma_t^\varphi(r_a(p)) = e^{itH_p^\varphi} r_a(p) e^{-itH_p^\varphi} \tag{11}$$

for every $p \in E$. Equation (11) can also be written in the form

$$i\hbar \frac{d}{dt}\sigma_t^{\varphi}(r_a(p)) = [r_a(p), H_p^{\varphi}] \tag{12}$$

where the Planck constant $\hbar$ is inserted to have the correspondence with quantum theory. This equation describes the state dependent evolution of random operators with respect to the parameter $t \in \mathbf{R}$ of the modular group. We can say that the pair $(\mathcal{M}, \varphi)$ is a *dynamical object* of our model. Equation (12) is a generalization of the Heisenberg equation of the usual quantum mechanics with the only difference that it now depends on the state φ. There exists the canonical way of getting rid of this dependence based on the following Connes–Nicodym–Radon construction [36]. Let $\mathcal{U} = \{u \in \mathcal{M} : uu^* = u^*u = 1\}$ be the unitary group of the algebra $\mathcal{M}$. Two automorphisms α_1 and α_2 of the von Neumann algebra $\mathcal{M}$ are said to be *inner equivalent* if there is an element $u \in \mathcal{U}$ such that

$$u\alpha_2(r) = \alpha_1(r)u$$

for $r \in \mathcal{M}$. The set of equivalence classes of this relation is called the group of *outer automorphisms* denoted by $Out(\mathcal{M})$. In general, the modular transformations σ_t^{φ} are not inner automorphisms of $\mathcal{M}$, but they canonically project onto the same one-parameter group in $Out(\mathcal{M})$ which is independent of the state φ. However, we have demonstrated in [22] that the von Neumann algebra $\mathcal{M}$ of our model is semifinite, and the Dixmier–Takesaki theorem [6] states that if $\mathcal{M}$ is semifinite then every state dependent modular evolution is inner equivalent to the trivial one. This means that the state independent "outer evolution" is trivial: there is a state independent time but it does not flow (or nothing happens in it). This once more demonstrates the radical character of the noncommutative regime of our model (in its present form)—it admits only a state dependent dynamics. In the following subsection, we shall show how this peculiarity is related to the concept of probability.

To sum up. There is an isomorphism $\mathrm{Inn}(\mathcal{A}) \to \mathcal{M}_0$ (by $\mathrm{ad}(a) \mapsto a \mapsto r_a$, see the next section), and the latter space can be completed to the von Neumann algebra which, together with a suitable state φ, forms a dynamical object of our model. We may thus say that (11) or (12) are natural dynamical equations for the quantum sector of our model, and that they can be traced back to the differential algebra $(\mathcal{A}, \mathrm{Inn}(\mathcal{A}))$.

7 Dynamical System

Mathematical structure of our model is encoded in the differential algebra $(\mathcal{A}, \mathrm{Der}(\mathcal{A}))$ where $\mathrm{Der}(\mathcal{A}) = V_1 \oplus V_2 \oplus V_3$. The Z-submodule $V = V_1 \oplus V_2$ is responsible for the gravitational sector, and the field equation for this sector assumes the form of the generalized eigenvalue equation (1). The submodule V_3 is responsible for the quantum sector, and the dynamics of this sector is given by "modular equation" (11) [resp. (12)]. Therefore, we can say that (1) and (11) [resp. (12)] form the "dynamical system" of our model. However, these two equations are of a very different character: (1) is classically geometric, and (11) is quantum probabilistic. Is there a possibility to make these equations "more unified"? Indeed, there is such a possibility. To show it, we must first prove the following lemma.

Lemma *The mapping $\pi : \mathcal{A} \to \mathcal{M}_0$, given by $\pi(a) = (\pi_p(a))_{p\in E}$, is an isomorphism of algebras.*

Proof The mapping π is an isomorphism since, for every $p \in E$, π_p is a representation of the algebra $\mathcal{A}$. Moreover, π is an injection. Indeed, let $r_a \in \mathcal{M}_0$ and $r_a = 0$. This means that $\pi(a) = 0$ for μ-almost all $p \in E$ where μ is a measure on E. Consequently, for μ-almost all $p \in E$ and all $\psi \in L^2(\Gamma^p)$ one has $\pi_p(a)\psi = 0$, i.e.,

$$(\pi_p(a)\psi)(p_1, p) = \int_{E_{\pi_M(p)}} a(p_1, p_2)\psi(p_2, p)dp_2 = \int_G a(p_1, pg)\psi(pg, p)dg = 0.$$

We have made the substitution $p_2 = pg$. Let $\psi(p_1, p) = \overline{a(p, p_1)}$. We have

$$(\pi_p(a)\psi)(p, p) = \int_G a(p, pg)\overline{a(p, pg)}dg = \int_G |a(p, pg)|^2 dg = 0.$$

This means that the support of $a_p := a(p, pg)$ is of zero measure and, consequently, the same is valid for a. But a is of class C^∞; therefore $a = 0$ which ends the proof.

Since the mapping $a \mapsto r_a$ is an isomorphism of algebras, every derivation $u : \mathcal{A} \to \mathcal{A}$ defines the derivation $\tilde{u} : \mathcal{M}_0 \to \mathcal{M}_0$ by

$$\tilde{u}(r_a) = r_{u(a)}.$$

This allows us to define the Einstein operator $\tilde{\mathbf{G}} : \tilde{V} \to \tilde{V}$, where $\tilde{V} \subset \mathrm{Der}(\mathcal{M}_0)$, by

$$\tilde{\mathbf{G}}(\tilde{u})(r_a) = r_{\mathbf{G}(u)a}.$$

This is valid only for $u \in V_1 \oplus V_2$ since, for $u \in V_3$, $\mathbf{G}(u)$ is not defined. Therefore, if $u \in V_1 \oplus V_2$ is an eigenvector of the Einstein operator $\mathbf{G}$ with the generalized eigenvalue τ then

$$\tilde{\mathbf{G}}(\tilde{u})(r_a) = r_{(\tau u)(a))} = \tau \cdot r_{u(a)} = \tau \cdot \tilde{u}(r_a).$$

We thus can write the "dynamical system" for our model in the following form

$$\tilde{\mathbf{G}}(\tilde{u}) = \tau \cdot \tilde{u} \tag{13}$$

for $u \in V_1 \oplus V_2$, and

$$\sigma_t^\varphi(r_a(p)) = e^{itH_p^\varphi} r_a(p) e^{-itH_p^\varphi} \tag{14}$$

for every $p \in E$ and a corresponding, in the unique manner, to the inner derivation $ad_a \in V_3$.

Since our model consists of the differential algebra $(\mathcal{A}, \mathrm{Der}(\mathcal{A}))$, its dynamical equation should constrain both $\mathcal{A}$ and $\mathrm{Der}(\mathcal{A})$. This is indeed the case: (13) is essentially for derivations, whereas (14) is for the algebra.

8 Dynamical Properties of the Quantum Sector

The proposed model has a remarkable unifying power. In this section, we show how in its mathematical structure dynamics, probability and thermodynamics are unified.

8.1 Noncommutative Probability

In classical probability theory, basic objects of study are random variables, i.e., measurable functions from a given probability space into the set of reals $\mathbf{R}$ (equipped with the Borel σ-algebra structure). With any such random variable X there is associated a probability measure $\mu_X(B)$ for any Borel set B. The measure μ_X is also called the *distribution* of X. In noncommutative probability theory [38, 39], random variables are replaced by operators on a Hilbert space $\mathcal{H}$. They are also called *noncommutative random variables*. Instead of working with the whole algebra $\mathcal{B}(\mathcal{H})$ of random operators on $\mathcal{H}$ one usually restricts to a subalgebra which is a von Neumann algebra. We recall that, by definition, it is a subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ containing the multiplicative unit of $\mathcal{B}(\mathcal{H})$ and closed under the adjoint operation and under taking limits in the weak topology on $\mathcal{B}(\mathcal{H})$, i.e., topology induced by the linear functionals $b \mapsto \langle b\xi, \eta\rangle$, $\xi, \eta \in \mathcal{H}$. Now, we must look for a suitable counterpart of the probability measure on $\mathcal{M}$. We need for it a kind of positivity and normalizability conditions. This is implemented by the concept of state on the von Neumann algebra $\mathcal{M}$. A linear functional $\varphi : \mathcal{M} \to \mathbf{C}$ is a *state* on $\mathcal{M}$ if it takes nonnegative values on positive operators on $\mathcal{M}$, and satisfies the condition $\varphi(1) = 1$. The pair $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is a von Neumann algebra and φ a state on $\mathcal{M}$, is called a *noncommutative probability space*; φ is called *probability measure* on $\mathcal{M}$. We shall additionally assume (as it is often done) that φ is a faithful and normal state on $\mathcal{M}$. *Faithful* means that φ does not annihilate any nonzero positive element of $\mathcal{M}$ [i.e., $\varphi(r) = 0$ implies $r = 0$ for any positive element $r \in \mathcal{M}$]. *Normal* means that if $r \in \mathcal{M}$ is the supremum of a monotonically increasing net $\{r_i\}$ in the collection of positive elements of $\mathcal{M}$ then $\varphi(r) = \text{supp}(r_i)$. The motivation for the above definition of noncommutative probability space comes from the fact that if $\mathcal{M}$ is a commutative von Neumann algebra, $\mathcal{M}$ is naturally isomorphic with the algebra of bounded measurable functions (modulo equality almost everywhere) on an interval.

We thus have an ensemble of noncommutative probability spaces $(\mathcal{M}, \varphi)_{\varphi \in F}$ where F denotes a collection of normal and faithful states on $\mathcal{M}$. As we have seen in the preceding subsection, each member $(\mathcal{M}, \varphi)$ of this ensemble is also a "dynamic object" defining the modular evolution σ_s^φ. In this context it seems natural that every noncommutative probability measure φ determines its own dynamics of random operators (for more see [22]). In this sense, two so far independent concepts are unified: every dynamics is probabilistic and every probability is dynamic.

8.2 Dynamics, Probability and Thermodynamics

For the physicist any probabilistic dynamics is inseparably linked with thermodynamic properties. To see that this is also the case in the noncommutative context let us first remember some theoretical concepts.

A state φ on the von Neumann algebra $\mathcal{M}$ is said to satisfy the *Kubo–Martin–Schwinger condition* (at inverse temperature β), or is simply said to be a *KMS state*, with respect to a one-parameter group $\{\sigma_s : s \in \mathbf{R}\}$ of automorphisms of $\mathcal{M}$ if, for each $A, B \in \mathcal{M}$, there exists a bounded continuous function on the strip $\{z \in \mathbf{C} : 0 \leq \operatorname{Im} z \leq 1\}$, $F : \{z \in \mathbf{C} : 0 \leq \operatorname{Im} z \leq 1\} \to \mathbf{C}$, which is analytic in the interior of the strip and satisfies the following conditions

$$F(s + \beta i) = \varphi(A\sigma_s(B))$$

and

$$F(s) = \varphi(\sigma_s(B)A),$$

for all s in $\mathbf{R}$. Here $\beta = 1/kT$ with k being the Boltzmann constant and T absolute temperature [36].

Let φ be a normal and faithful state, and σ_t^{φ}, $t \in \mathbf{R}$, the modular group. Then φ is a KMS state with respect to σ_t^{φ} and satisfies the condition $\varphi \circ \sigma_t^{\varphi} = \varphi$. Moreover, such a modular group σ_t^{φ} is uniquely determined ([7], Sect. 2.5).

In quantum field theoretical statistical mechanics, KMS states are interpreted as thermal equilibrium states (at inverse temperature β). Rovelli argued that also in the relativistic context equilibrium states can be characterized as faithful states on the algebra of observables whose modular group is σ_s^{φ} [27, 34]. If we adopt this interpretation, we can claim that in the noncommutative regime of our model dynamics, probability and at least some aspects of thermodynamics are unified in the same mathematical structure.

9 Random Dynamics of the Closed Friedman Universe

To illustrate the random behavior in our model, let us return to the noncommutative version of the closed Friedman universe. For simplicity we consider the two dimensional case $M = [0, T] \times S^1$. It is obvious that the random evolution in this world model is expected to occur at its earliest and latest phases in neighborhoods of its initial and final singularities.

The representation of the algebra $\mathcal{A}$, $\pi_p : \mathcal{A} \to \mathcal{B}(L^2(\Gamma^p))$, where $\Gamma^p = \{(\eta, \chi, \lambda_1, \lambda) : \lambda_1 \in \mathbf{R}\}$ for $p = (\eta, \chi, \lambda))$, is now given by

$$(\pi_p(a)\psi)(\lambda_1) = \int_{\mathbf{R}} a(\eta, \chi, \lambda_1, \lambda_2)\psi(\eta, \chi, \lambda_2, \lambda)d\lambda_2,$$

$a \in \mathcal{A}$, $\psi \in L^2(\Gamma^p)$, and λ is fixed.

The isomorphisms $I_p : L^2(\mathbf{R}) \to L^2(\Gamma^p)$ are given by

$$(I_p(\psi_0))(\eta, \chi, \lambda_1, \lambda) = \psi_0(\lambda_1)$$

for $\psi_0 \in L^2(\mathbf{R})$. In this case, the regular representation assumes the form

$$(\tilde{\pi}_p(a)\psi_0)(\lambda_1) = \int_{\mathbf{R}} a(\eta, \chi, \lambda_1, \lambda_2)\psi_0(\lambda_2)d\lambda_2 = \int_{\mathbf{R}} a_{\eta,\chi}(\lambda_1, \lambda_2)\psi_0(\lambda_2)d\lambda_2.$$

The operator $\tilde{\pi}_p(a)$ is Hermitian if $a_{\eta,\chi}(\lambda_2, \lambda_1) = \overline{a_{\eta,\chi}(\lambda_1, \lambda_2)}$, $\lambda_1, \lambda_2 \in \mathbf{R}$, for every $(\eta, \chi) \in M$. We have the norm ess $\sup(\|\tilde{\pi}_p(a)\|) < \infty$. Therefore, $(\tilde{\pi}_p(a))_{p\in E}$ are random operators. The algebra $\mathcal{M}_0$ of equivalence classes (modulo equality everywhere) of bounded random operators is of the form $\mathcal{M}_0 = \{E \ni p \mapsto \tilde{\pi}_p(a) \in \mathcal{B}(L^2(\mathbf{R})) : a \in \mathcal{A}\}$. It can be shown [33] that $\mathcal{M}_0$ generates the von Neumann algebra

$$\mathcal{M} \simeq L^{\infty}(M, \mathcal{B}(L^2(\mathbf{R})).$$

Let $A = (\tilde{\pi}_p(a))_{p\in E}$. Let us also notice that it is enough to define the states on $\mathcal{M}_0$. On the strength of Proposition B.1 of [33] such normal states are of the form

$$\varphi(A) = \int_{M\times\mathbf{R}\times\mathbf{R}} \mathrm{a}(\eta, \chi, \lambda_1, \lambda_2)\rho(\eta, \chi, \lambda_1, \lambda_2)d\eta d\chi d\lambda_1 d\lambda_2 \tag{15}$$

where ρ is the density function. It must be nonnegative, Hermitian and integrable with the corresponding integral equal to 1. To be faithful it must satisfy the condition:

$\rho(\eta, \chi, \lambda_1, \lambda_2) > 0$ (modulo zero measure subsets). Of course, there is one-to-one correspondence between φ and ρ.

The important fact is that functional (15) is well defined also in the presence of the initial and final singularities. In the closed Friedman world model both these singularities are malicious, and consequently the outer center Z consists only of constant functions [15, 16, 21], but this fact has no influence on the form of functional (15). Therefore, we can say that $\varphi(A)$ does not "feel" any singularity. Let us also notice that the functional $\varphi(A)$ prolongs well to Z; namely, if $f \in Z$, one has

$$\varphi(A) = k \int_{M \times \mathbf{R} \times \mathbf{R}} \rho(\eta, \chi, \lambda_1, \lambda_2) d\eta d\chi d\lambda_1 d\lambda_2 = k$$

where k is a constant value of f. This means that from the macroscopic point of view $\varphi(A)$ is constant.

On the strength of the Tomita–Takesaki theorem, the functional φ (which is normal and faithful) determines a modular group σ_t^ϕ, $t \in \mathbf{R}$ of automorphisms of the von Neumann algebra $\mathcal{M}$. In terms of this group one can define the (state dependent) dynamics of random operators $(\tilde{\pi}_p(a))_{p \in \bar{E}}$, $a \in \bar{\mathcal{A}}$. Since the functional φ does not feel singularities, they are irrelevant for the dynamics of the Friedman model on its fundamental level. They appear only in the process of taking the ratio $\bar{M} = \bar{E}/G$ when space-time M emerges out of the noncommutative regime; bars over M and E denote here suitable completions of M and E, correspondingly (for the analysis of this process see [15, 16, 19]).

10 Transition to General Relativity and Quantum Mechanics

It is clear that to go from our model to general relativity one must "restrict" the algebra $\mathcal{A} = C^\infty(\Gamma, \mathbf{C})$ to its "outer center" $Z = \pi_M^*(C^\infty(M))$ which, being isomorphic to $C^\infty(M)$, naturally reproduces the usual spacetime geometry. It is interesting that this can also be done with the help of the following "averaging" procedure. Let $\tilde{\mathcal{A}}$ be the extension of the algebra $\mathcal{A}$

$$\tilde{\mathcal{A}} = \{a \in C^\infty(\Gamma, \mathbf{C}) : \forall x \in M, a|\Gamma_x \in C_c^\infty(\Gamma_x, \mathbf{C})\}$$

where $\Gamma_x = E_x \times G$. Let further $\tilde{a}$ be the function defined on $E \times G \times G$, corresponding to a, defined in the following way: $\tilde{a}(p_0, g_1, g_2) = a(p_0 g_1, g_1^{-1} g_2)$. Then the "averaging" of a is defined to be

$$\langle a \rangle = (\mathrm{Tr}\, a)(x) = \int_G \tilde{a}(p_0, g, g) dg. \tag{16}$$

It is clear that $\langle a \rangle \in Z$ which is isomorphic to the algebra $C^\infty(M)$, and in terms of this algebra general relativity can be reconstructed [12–14].

The transition to quantum mechanics is even more interesting. If a is a Hermitian element of the algebra $\mathcal{A}$ then $\pi_p(a)$ is a Hermitian element of $(\mathcal{B}(\mathcal{H}^p))$ (since π_p is a $*$-representation of the algebra $\mathcal{A}$). A random operator $r_a(p) = \pi_p(a)$ is Hermitian if $(r_a(p)\psi, \varphi) = (\psi, r_a(p)\varphi)$. Moreover, it is a compact operator since a has the compact support. On the strength of the spectral theorem for Hermitian compact operators in a separable Hilbert space, there exists in $\mathcal{H}^p$ an orthonormal countable Hilbert basis of eigenvectors $\{\psi_i\}_{i \in I}$ of the Hermitian operator $r_a(p)$, and we can write its eigenvalue equation as

$$r_a(p)\psi_i(p) = \lambda_i(p)\psi_i(p)$$

for every $p \in E$. Here $\lambda_i : E \to \mathbf{R}$ is a generalized eigenvalue of the operator r_a. However, every measurement is always done in a given local reference frame $p \in E$, and when such a measurement has been done the generalized eigenvalue λ_i collapses to the eigenvalue $\lambda_i(p)$. Let us look deeper into the mechanism of this collapse. Each act of measurement, performed at p, defines the isomorphism $I_p^{-1} : \mathcal{H}^p \to L^2(G)$ of Hilbert spaces (see, Sect. 4.1) which transfers the algebra of random operators into the usual algebra of operators on the Hilbert space $L^2(G)$. In this way, one obtains the usual quantum mechanics (on the group G).

For instance, let us apply the mapping I_p^{-1} to the left hand side of (12), and the mapping I_p to its right hand side. By doing so, we obtain the usual Heisenberg equation for the evolution of $a \in \mathcal{A}$

$$\frac{d}{dt}\tilde{\pi}(a(t)) = i[\tilde{H}^{\varphi}, \tilde{\pi}(a(t))]$$

where $\tilde{\pi}(a) = I_p^{-1} \circ \pi_p \circ I_p$ and $\tilde{H}^{\varphi} = I_p^{-1} \circ H_p^{\varphi} \circ I_p$. The only difference as compared with the usual Heisenberg equation is that the above equation depends on the state φ. In more realistic models, to which the Connes–Nikodym–Radon construction applies, even this difference will disappear [23].

In the light of the above analysis, the usual quantum mechanics is but a theory of measurement within the larger structure of our model. When the act of measurement is performed, the larger structure collapses to its substructure known as quantum mechanics.

11 Perspectives

We do not claim that the model presented in this work should be regarded as a concurrence with respect to theories like superstring theory or quantum loop theory. First, it is not advanced enough and, second, we treat it rather as a mean to deepen our understanding of conceptual subtleties that are to be met along the road leading to the unification of physics. It is not impossible that some elements of this model, or of its future more mature forms, could be incorporated into better known approaches.

The noncommutative version of the closed Friedman world model, presented in this work, is only a "test model", but it exhibits a remarkable property. Although in the original field equation no matter term was explicitly included, the correct components of the energy-momentum tensor (density and pressure) are obtained as generalized eigenvalues of the Einstein operator. This effect can be regarded as essentially mitigating the strong dichotomy between geometry and matter inherent in the usual Einstein field equation.

Another interesting problem related to the search for a fundamental theory is whether the initial (or final) singularity will survive such a revolution. Usually, either "yes" or (more often) "no" answers are given to this question. Our model discloses the third possibility. Simple calculations for a closed Friedman world model show that the random character of dynamics on the fundamental level makes the question concerning the singularities irrelevant. Singularities emerge from the noncommutative regime together with the macroscopic spacetime. This result remains in agreement with our previous works on classical singularities with the help of noncommutative methods [15, 16, 19].

Although the conceptual structure of our model seems esthetically satisfactory in many respects, we are aware of various its limitations; some of them could be overcome by enriching the architecture of the model. This could be done in many ways, perhaps the most obvious would be by taking into account the bialgebraic (or Hopf algebraic) structure of the groupoid algebra (some coalgebra structure has been taken into account in discussing observables of the model [21]). The obvious next step to do is the elaboration of quantum field theoretical aspects of the model (spinor bundles, Dirac's operator, etc.) together with the gauge theoretic approach. Some preliminary work in this direction is under way.

References

1. Alicki, R., Fanes, M.: Rep. Math. Phys. **55**, 47 (2005)
2. Carmeli, M.: Group Theory and General Relativity. McGraw–Hill, New York (1977)
3. Chamseddine, A.H., Felder, G., Fröhlich, J.: Commun. Math. Phys. **155**, 205 (1993)
4. Chamseddine, A.H.: Phys. Lett. B **504**, 33 (2001)
5. Chamseddine, E.H.: Phys. Rev. D **69**, 024015 (2004)
6. Connes, A.: Noncommutative Geometry. Academic, New York (1994)
7. Connes, A., Rovelli, C.: Class. Quantum Gravity **11**, 2899 (1994)
8. Dixmier, J.: Von Neumann Algebras. North-Holland, Amsterdam (1981)
9. Dodson, C.T.J.: Int. J. Theor. Phys. **17**, 389 (1978)
10. Dubois–Violette, M.: Comptes Rendu Acad. Sci. Paris **307**, 403 (1988)
11. Dubois–Violette, M., Michor, P.W.: Comptes Rendu Acad. Sci. Paris **319**, 927 (1994)
12. Geroch, R.: Commun. Math. Phys. **26**, 271 (1972)
13. Heller, M.: Int. J. Theor. Phys. **31**, 272 (1992)
14. Heller, M., Sasin, W.: Int. J. Theor. Phys. **34**, 387 (1995)
15. Heller, M., Sasin, W.: J. Math. Phys. **36**, 3644 (1995)
16. Heller, M., Sasin, W.: Gen. Relativ. Gravit. **31**, 555 (1999)
17. Heller, M., Sasin, W., Lambert, D.: J. Math. Phys. **38**, 5840 (1997)
18. Heller, M., Sasin, W., Odrzygóźdź, Z.: Int. J. Theor. Phys. **38**, 1619 (1999)
19. Heller, M., Odrzygóźdź, Z., Pysiak, L., Sasin, W.: Int. J. Theor. Phys. **42**, 427 (2003)
20. Heller, M., Odrzygóźdź, Z., Pysiak, L., Sasin, W.: Gen. Relativ. Gravit. **36**, 111 (2004)
21. Heller, M., Odrzygóźdź, Z., Pysiak, L., Sasin, W.: Gen. Relativ. Gravit. **37**, 541 (2005)
22. Heller, M., Pysiak, L., Sasin, W.: Int. J. Theor. Phys. **44**, 619 (2005)
23. Heller, M., Pysiak, L., Sasin, W.: J. Math. Phys. **46**, 122501 (2005)
24. Madore, J.: An Introduction to Noncommutative Differential Geometry and Its Physical Applications. 2nd edn. Cambridge University Press, Cambridge (1999)
25. Madore, J., Mourad, J.: J. Math. Phys. **39**, 424 (1998)
26. Madore, J., Saeger, L.A.: Class. Quantum Gravity **15**, 811 (1998)
27. Martinetti, P., Rovelli, C.: Class. Quantum Gravity **22**, 4919 (2003)
28. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. Freeman, San Francisco (1973)
29. Moffat, J.W.: Phys. Lett. **B 491**, 345 (2000)
30. Murphy, G.J.: C*-algebras and Operator Theory. Academic, New York (1990)
31. Paterson, A.L.: Groupoids, Inverse Semigroups and Their Operator Algebras. Birkhäuser, Boston (1999)
32. Pysiak, L.: Demonstr. Math. **37**, 661 (2004)
33. Pysiak, L.: Time flow in a noncommutative regime. Int. J. Theor. Phys. **46**, 17 (2007)
34. Rovelli, C.: Class. Quantum Gravity **10**, 1549 (1993)
35. Sitarz, A.: Class. Quantum Gravity **11**, 2127 (1994)
36. Sunder, V.S.: An Invitation to von Neumann Algebras. Springer, New York (1987)
37. Szabo, R.J.: Class. Quantum Gravity **23**, R199 (2006)
38. Voiculescu, D.V.: Symmetries of some reduced free product C^*-algebras. In: Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Mathematics, vol. 1132, Springer, Berlin (1985)
39. Voiculescu, D.V., Dykema, K., Nica, A.: Free Random Variables. CRM Monograph Series, vol. 1, American Mathematical Society, Providence (1992)
40. Wheeler, J.A.: Pregeometry: motivations and prospects. In: Marlow, A.R. (ed.) Quantum Theory and Gravitation, Academic, New York (1980)